Appl. Math. Mech. -Engl. Ed., 2008, **29**(12):1535–1552
DOI 10.1007/s10483-008-1202-x

Applied Mathematics and Mechanics (English Edition)

Solution of two-dimensional scattering problem in piezoelectric/piezomagnetic media using a polarization method *

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Abstract Using a polarization method, the scattering problem for a two-dimensional inclusion embedded in infinite piezoelectric/piezomagnetic matrices is investigated. To achieve the purpose, the polarization method for a two-dimensional piezoelectric/piezomagnetic "comparison body" is formulated. For simple harmonic motion, kernel of the polarization method reduces to a 2-D time-harmonic Green's function, which is obtained using the Radon transform. The expression is further simplified under conditions of low frequency of the incident wave and small diameter of the inclusion. Some analytical expressions are obtained. The analytical solutions for generalized piezoelectric/piezomagnetic anisotropic composites are given followed by simplified results for piezoelectric composites. Based on the latter results, two numerical results are provided for an elliptical cylindrical inclusion in a PZT-5H-matrix, showing the effect of different factors including size, shape, material properties, and piezoelectricity on the scattering cross-section.

Key words scattering, piezoelectric/piezomagnetic material, polarization method, dynamic Green's function, two-dimensional problem, Radon transform, anisotropic material

Chinese Library Classification O347.4, O302
2000 Mathematics Subject Classification 74J20

Introduction

Wave scattering by inclusions embedded in composites has long been the subject of investigation by many researchers. Pao[1] made a detailed discussion about diffraction of elastic waves and its relationship with dynamic stress concentration. However, his discussion was limited to isotropic materials. On the other hand, Auld's work[2] about acoustic waves in solids did pay some attention to wave propagation in general anisotropic materials. However, scattering was not considered as a topic in the book. Barnett[3] pointed out that following the method first introduced by Stroh[4], discussion of wave propagation in solids with general anisotropy could

* Received Sept. 3, 2008 / Revised Sept. 18, 2008
Project supported by the National Natural Science Foundation of China (Nos. 10732100, 10572155), the Science and Technology Planning Project of Guangdong Province of China (No. 2006A11001002), and the Ph. D. Programs Foundation of Ministry of Education of China (No. 20063000041111179)
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be treated in a quite simple fashion. Indeed, speaking of two-dimensional problems with general anisotropy, whether for static or for dynamic problem, people would always prefer to try using Stroh's formalism, for its mathematic elegance and conciseness. Stroh's method is successful in finding the Rayleigh wave for half-space, which can be considered as the simplest form of scattering. Wu[5] extended Stroh's formalism to treat the so-called "self-similar" problem, which was another success in treating dynamic problems.

However, except for the simplest case of the Rayleigh wave, Stroh's formalism does not give exercisable result for general simple harmonic motions: substitution of functions with the form $f(x_1+px_2)\mathrm{e}^{\mathrm{i}wt}$ into the dynamic equilibrium equation does not give simple eigenfunctions anymore. On the other hand, the extended Stroh's method formulated by Wu[5] can only treat scattering problems caused by inhomogeneities with boundaries on $x_2=0$. For a more general two-dimensional scattering problem, until now, Stroh's formalism cannot give satisfying results. To sum up, up to now, no solution is available to the two-dimensional scattering problem caused by simple harmonic waves colliding on an inhomogeneity with a rather general shape, let alone the discussion in piezoelectric/piezomagnetic materials.

As the preferred Stroh's method does not promise any satisfying results, the two-dimensional scattering problem with inhomogeneities of arbitrary shape has been put into a rather embarrassing condition. Besides Stroh's formalism, Willis[6] provided another effective approach–polarization method to deal with the three-dimensional scattering problem, which was formulated as integral equations of Green's function. Ma and Wang[7] developed this method and adopted several concepts introduced by the classical method of Eshelby[8] to treat the scattering problem caused by an ellipsoidal inclusion in infinite anisotropic piezoelectric matrices.

In this paper, we combine Willis' idea[6] with some other powerful analytical skills, such as the Radon transform and the Residue theorem in complex methods, to treat two-dimensional scattering problems in piezoelectric/piezomagnetic composites. Besides an integral solution for the general problem, some exact analytical results are obtained for the first time under certain simplification.

1 Basic equations

Using the extended Barnett and Lothe notation[9] and quoting Pan's expression[10], the equation of equilibrium for the coupled magneto-electro-elastic field of any media with general anisotropy can be expressed as

$$C_{iJKl}u_{K,li}+f_J=\rho_{JK}\ddot{u}_K, \tag{1}$$

where

$$C_{iJKl}=\begin{cases}C_{ijkl}, & J,K=1,2,3;\\ e_{lij}, & J=1,2,3,\quad K=4;\\ e_{ikl}, & J=4,\quad K=1,2,3;\\ q_{lij}, & J=1,2,3,\quad K=5;\\ q_{ikl}, & J=5,\quad K=1,2,3;\\ -\lambda_{il}, & J=4,\quad K=5 \text{ or } J=5,\quad K=4;\\ -\varepsilon_{il}, & J,K=4;\\ -\mu_{il}, & J,K=5;\end{cases} \tag{2}$$

and

$$u_J=\begin{cases}u_j, & J=1,2,3;\\ \phi, & J=4;\\ \varphi, & J=5;\end{cases}\qquad f_J=\begin{cases}f_j, & J=1,2,3;\\ -f_{\mathrm{e}}, & J=4;\\ -f_{\mathrm{m}}, & J=5;\end{cases}\qquad \rho_{JK}=\begin{cases}\delta_{JK}\rho, & J,K=1,2,3;\\ 0, & \text{other cases.}\end{cases} \tag{3}$$

$C_{ijkl}, \varepsilon_{ij}$ and μ_{ij} are the elastic, dielectric, and magnetic permeability tensors, respectively. e_{ijk}, q_{ijk} and λ_{ij} are the piezoelectric, piezomagnetic, and magnetoelectric coefficients, respectively. u_j, ϕ and φ are the elastic displacement, electric potential, and magnetic potential, respectively. $f_j, f_{\rm e}$ and $f_{\rm m}$ are the body force, electric charge, and electric current (or called the magnetic charge as compared to the electric charge), respectively. δ_{JK} denotes the Kronecher tensor and ρ is the density. $\ddot{u}_J$ indicates the second-order derivative of u_J with respect to time. The extended elastic coefficient tensor C_{iJKl} in Eq. (1) relates the extended strains to the extended stresses by the constitutive relationship,

$$\sigma_{iJ} = C_{iJKl}\gamma_{Kl}, \tag{4}$$

where the extended stresses and strains are defined by

$$\sigma_{iJ} = \begin{cases} \sigma_{ij}, & J = 1,2,3; \\ D_i, & J = 4; \\ B_i, & J = 5; \end{cases} \qquad \gamma_{Ij} = \begin{cases} \gamma_{ij}, & I = 1,2,3; \\ -E_j, & I = 4; \\ -K_j, & I = 5. \end{cases} \tag{5}$$

In Eq. (5), σ_{ij}, D_i and B_i are the stress, electric displacement, and magnetic induction (i.e., magnetic flux), respectively; γ_{ij}, E_j and K_j are the strain, electric field and magnetic field, respectively. It is observed that various uncoupled cases (i.e., purely elasticity, piezoelectricity, and piezomagneticity) can be reduced from Eqs. (1)–(5) by setting the appropriate coefficients to zero. It is further noticed that the following symmetry relationship holds:

$$\begin{cases} C_{ijkl} = C_{jikl} = C_{klij}, \\ e_{kji} = e_{kij}, \quad q_{kji} = q_{kij}, \\ \varepsilon_{ij} = \varepsilon_{ji}, \quad \lambda_{ij} = \lambda_{ji}, \quad \mu_{ij} = \mu_{ji}. \end{cases} \tag{6}$$

Finally, the extended strains and displacements are related by the geometric equation,

$$\begin{cases} \gamma_{ij} = \dfrac{1}{2}(u_{i,j} + u_{j,i}), \\ E_i = -\phi_{,i}, \quad H_i = -\varphi_{,i}. \end{cases} \tag{7}$$

2 Two-dimensional polarization method for composites

Consider an infinite piezoelectric/piezomagnetic body composed of matrices with generalized elastic moduli C^0_{iJKl} and density ρ^0_{JK} and a two-dimensional inclusion (e.g., cylinder with infinite length) embedded in matrices occupying Ω with generalized elastic moduli $C^{\rm I}_{iJKl}$ and density $\rho^{\rm I}_{JK}$. Set the coordinate so that the x_3-axis coincides with the length of the inclusion. Equation (1) becomes

$$(C^0_{iJKl}u_{K,l})_{,i} + f_J + \tau_{iJ,i} - \dot{\pi}_J = \rho^0_{JK}\ddot{u}_K, \tag{8}$$

where

$$\tau_{iJ} = \Delta C_{iJKl}u_{K,l}, \quad \pi_J = \Delta\rho_{JK}\dot{u}_K, \tag{9}$$

$$\Delta C_{iJKl} = (C^{\rm I}_{iJKl} - C^0_{iJKl})H(\boldsymbol{x}), \quad \Delta\rho_{JK} = (\rho^{\rm I}_{JK} - \rho^0_{JK})H(\boldsymbol{x}), \tag{10}$$

and

$$H(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \in \Omega, \\ 0, & \boldsymbol{x} \notin \Omega. \end{cases} \tag{11}$$

In Eqs. (8)–(11) and derivation in Section 2, $\boldsymbol{x}=(x_1,x_2)$, and the capital subscripts (e.g., J,K) run from 1 to 5, the lowercase subscripts (e.g., i,l) run from 1 to 2.

Now consider a "comparison body", which is homogeneous and takes the material of the matrices. The equilibrium equation of this body is

$$(C^0_{iJKl}u_{K,l})_{,i}+f_J=\rho^0_{JK}\ddot{u}_K. \tag{12}$$

The adjoint problem of Eq. (12), corresponding to the adjoint operators C^*_{iJKl} and ρ^*_{JK}, is governed by the following equations:

$$(C^*_{lKJi}v_{J,i})_{,l}+F_K=\rho^*_{KJ}\ddot{v}_J, \tag{13}$$

where F_K is the extended body force of the adjoint problem for the field $\boldsymbol{v}$. The adjoint operators are defined by

$$\int_0^\infty\int_S(v_{J,i}C^0_{iJKl}u_{K,l}-u_{K,l}C^*_{lKJi}v_{J,i})dSdt=0, \tag{14}$$

$$\int_0^\infty\int_S(\dot{v}_J\rho^0_{JK}\dot{u}_K-\dot{u}_K\rho^*_{KJ}\dot{v}_J)dSdt=0. \tag{15}$$

By use of Gauss's theorem, Eqs. (12)–(15) lead to the identity,

$$\begin{aligned}&\int_0^\infty\int_{\partial S}[v_JC^0_{iJKl}u_{K,l}n_i-u_KC^*_{lKJi}v_{J,i}n_l]d\lambda dt+\int_0^\infty\int_S(v_JF_J+u_KF_K)dSdt\\ =&\int_S(v_J\rho^0_{JK}\dot{u}_K-u_K\rho^*_{KJ}\dot{v}_J)|^\infty_{t=0}dS.\end{aligned} \tag{16}$$

Green's function $\boldsymbol{G}$ and its components for the comparison body satisfy

$$(C^0_{iJKl}G_{KP,l})_{,i}+\delta_{JP}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')=\rho^0_{JK}\ddot{G}_{KP}, \tag{17}$$

with homogeneous initial and boundary conditions. The two indexes of $G_{KP}(\boldsymbol{x}-\boldsymbol{x}',t-t')$ denote the component of the extended Green's displacement and the direction of the extended point force. And let $\boldsymbol{G}^*$ be the adjoint Green's function, whose components follow

$$(C^*_{lKJi}G^*_{JQ,i})_{,l}+\delta_{KQ}\delta(\boldsymbol{x}-\boldsymbol{x}'')\delta(t-t'')=\rho^*_{KJ}\ddot{G}^*_{JQ} \tag{18}$$

with the corresponding adjoint boundary conditions. According to Eq. (16), we have

$$G^*_{QP}(\boldsymbol{x}'-\boldsymbol{x}'',t'-t'')=G_{PQ}(\boldsymbol{x}''-\boldsymbol{x}',t''-t'). \tag{19}$$

Equation (19) shows that the adjoint Green's function $\boldsymbol{G}^*$ may be obtained directly from $\boldsymbol{G}$, which is more convenient for derivation. Application of Eq. (16) to Eq. (8) yields

$$u_Q(\boldsymbol{x}'',t'')=-\int\int_S[G^*_{JQ,i}(\boldsymbol{x}''-\boldsymbol{x},t''-t)\tau_{iJ}(\boldsymbol{x},t)-\dot{G}^*_{JQ}(\boldsymbol{x}''-\boldsymbol{x},t''-t)\pi_J]dSdt+u^0_Q(\boldsymbol{x}'',t''), \tag{20}$$

in which

$$\begin{aligned}u^0_Q(\boldsymbol{x}'',t'')=&-\int\int_S[G^*_{JQ}(\boldsymbol{x}''-\boldsymbol{x},t''-t)f_J(\boldsymbol{x},t)]dSdt\\&-\int\int_{\partial S}[u_KC^*_{lKJi}G^*_{JQ,i}n_l-G^*_{JQ}(C^0_{iJKl}u_{K,l}+\pi_{iJ})n_i]d\lambda dt\\&+\int_S\{G^*_{JQ}(\boldsymbol{x}''-\boldsymbol{x},t'')[\rho^0_{JK}u_{K,l}(\boldsymbol{x},0)+\pi_J(\boldsymbol{x},0)]\\&-u_K(\boldsymbol{x},0)\rho^*_{KJ}G^*_{JQ}(\boldsymbol{x}''-\boldsymbol{x},t'')\}dS.\end{aligned} \tag{21}$$

It should be emphasized that Eq. (20) is valid only when momentum rather than velocity is given as the initial condition. Comparing Eqs. (20) and (21) with Willis's results[6] for the inhomogeneous anisotropic case, it follows that the present solution can be reduced to the piezoelectric, piezomagnetic, or pure elastic case when the corresponding moduli vanish. Symbolically, there is

$$\boldsymbol{u} = -\boldsymbol{N}\boldsymbol{\tau} - \boldsymbol{M}\boldsymbol{\pi} + \boldsymbol{u}^0, \tag{22}$$

where

$$(N\tau)_Q(\boldsymbol{x}, t) = \int\int_S N_{QiJ}(\boldsymbol{x} - \boldsymbol{x}', t - t')\tau_{iJ}(\boldsymbol{x}', t')dS'dt', \tag{23}$$

$$(M\pi)_Q(\boldsymbol{x}, t) = \int\int_S M_{QJ}(\boldsymbol{x} - \boldsymbol{x}', t - t')\pi_J(\boldsymbol{x}', t')dS'dt', \tag{24}$$

and

$$N_{QiJ}(\boldsymbol{x} - \boldsymbol{x}', t - t') = \frac{\partial G^*_{JQ}(\boldsymbol{x}' - \boldsymbol{x}, t' - t)}{\partial x'_i} = \frac{\partial G_{QJ}(\boldsymbol{x} - \boldsymbol{x}', t - t')}{\partial x'_i}, \tag{25}$$

$$M_{QJ}(\boldsymbol{x} - \boldsymbol{x}', t - t') = -\frac{\partial G^*_{JQ}(\boldsymbol{x}' - \boldsymbol{x}, t' - t)}{\partial t'} = \frac{\partial G_{QJ}(\boldsymbol{x} - \boldsymbol{x}', t - t')}{\partial t'}. \tag{26}$$

Substitution of Eq. (22) into Eq. (9) gives

$$(\Delta C)^{-1}_{iJQl}\tau_{iJ} + (N_{\boldsymbol{x}}\tau)_{Ql} + (M_{\boldsymbol{x}}\pi)_{Ql} = u^0_{Q,l}, \tag{27}$$

$$(\Delta\rho)^{-1}_{JQ}\pi_J + (N_t\tau)_Q + (M_t\pi)_Q = \dot{u}^0_Q, \tag{28}$$

where

$$(N_x)_{QliJ} = \frac{\partial^2 G_{QJ}}{\partial x_l \partial x'_i}, \quad (M_x)_{QlJ} = \frac{\partial^2 G_{QJ}}{\partial x_l \partial t'}, \tag{29}$$

$$(N_t)_{QiJ} = \frac{\partial^2 G_{QJ}}{\partial t \partial x'_i}, \quad (M_t)_{QJ} = \frac{\partial^2 G_{QJ}}{\partial t \partial t'}. \tag{30}$$

Consider a two-dimensional incident wave taking place in the body, whose expression is given by

$$\boldsymbol{u}^0 = \boldsymbol{a}\exp\{-\mathrm{i}[k_0(\boldsymbol{n}^0 \cdot \boldsymbol{x}) + wt]\}, \tag{31}$$

where $\boldsymbol{n}^0 = [n_1, n_2]^{\mathrm{T}}$ is a unit vector. The polarization $\boldsymbol{a}$ and the wave number k_0 satisfy

$$\Big[\boldsymbol{Q}n_1^2 - \boldsymbol{\rho}\frac{w^2}{k_0^2} + (\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}})n_1 n_2 + \boldsymbol{T}n_2^2\Big]\boldsymbol{a} = 0, \tag{32}$$

where the superscript T stands for matrix transpose, and $\boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{T}$ are defined respectively as

$$\boldsymbol{Q} = [C_{1JK1}], \quad \boldsymbol{R} = [C_{1JK2}], \quad \boldsymbol{T} = [C_{2JK2}], \tag{33}$$

and $\boldsymbol{\rho}$ is defined as

$$\boldsymbol{\rho} = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{34}$$

When such a problem is considered, the scattered wave caused by the inclusion will depend on time t through a factor $\exp(-\mathrm{i}wt)$. Correspondingly, time-reduced versions of operators $\boldsymbol{N}$ and $\boldsymbol{M}$ are required. They are, on the other hand, obtained from the two-dimensional time-reduced Green's function for dynamics.

3 Two-dimensional time-harmonic Green's function for dynamics

The two-dimensional time-harmonic dynamic Green's function for piezoelectric/piezomagnetic solids with general anisotropy is first obtained in this section. First, let a generalized time-harmonic line force apply along the x_3-axis in the x_Q-direction, and time $t = -\infty$. We get

$$f_K(\boldsymbol{x}, t) = \delta_{KQ}\delta(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}wt}, \tag{35}$$

where $\boldsymbol{x}$ is in plane (x_1, x_2). It follows that the generalized displacements in the solids are also time-harmonic, which can be written as

$$u_Q(\boldsymbol{x}, t) = G_{QJ}(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}wt}. \tag{36}$$

Substitution of Eqs. (35) and (36) into Eq. (1) yields

$$(L_{JK}(\partial) + \rho_{JK}w^2)G_{QJ} = -\delta_{QK}\delta(\boldsymbol{x}), \tag{37}$$

where

$$L_{JK}(\partial) = C_{1JK1}\frac{\partial^2}{\partial x_1^2} + (C_{1JK2} + C_{2JK1})\frac{\partial^2}{\partial x_1 \partial x_2} + C_{2JK2}\frac{\partial^2}{\partial x_2^2}. \tag{38}$$

From Eq. (37), it is obvious that the well-known Stroh formalism cannot be applied to this problem. If we apply $G_{QJ} = G_{QJ}(x_1+px_2)$ to Eq. (37), we find that the equation is difficult and does not promise a solution. It turns out that because of the existence of the term $\rho_{JK}w^2G_{QJ}$, the eigenvalue equation in Stroh's formalism is no longer available. We have to find another method to deal with the problem. An application of the two-dimensional Radon transform defined by Eq. (A1) to both sides of Eq. (37) gives

$$(L_{JK}(\boldsymbol{n})\frac{\partial^2}{\partial s^2} + \rho_{JK}w^2)\hat{G}_{QJ}(s) = -\delta_{QK}\delta(s), \tag{39}$$

where

$$L_{JK}(\boldsymbol{n}) = C_{1JK1}n_1^2 + (C_{1JK2} + C_{2JK1})n_1n_2 + C_{2JK2}n_2^2. \tag{40}$$

According to Eq. (6), $L_{JK} = L_{KJ}$. In Eq. (39), the density tensor can be expended as

$$[\rho_{JK}] = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{41}$$

Equation (39) can be decomposed as follows:

$$\Big(L_{jk}(\boldsymbol{n})\frac{\partial^2}{\partial s^2} + \rho\delta_{jk}w^2\Big)\hat{G}_{qj}(s) + L_{4k}(\boldsymbol{n})\frac{\partial^2\hat{G}_{q4}(s)}{\partial s^2} + L_{5k}(\boldsymbol{n})\frac{\partial^2\hat{G}_{q5}(s)}{\partial s^2} = -\delta_{qk}\delta(s), \tag{42}$$

where most of the results are omitted here. These equations are of the similar form with Eq. (42) and too lengthy to be presented here. In Eq. (42) and later discussion, the lowercase subscripts (e.g. j, k) run from 1 to 3. We also have equations with the following form:

$$\frac{\partial^2\hat{G}_{q4}}{\partial s^2} = (L_{55}^{-1}L_{45} - L_{54}^{-1}L_{44})^{-1}(L_{54}^{-1}L_{j4} - L_{55}^{-1}L_{j5})\frac{\partial^2\hat{G}_{qj}}{\partial s^2}. \tag{43}$$

Substitution of Eq. (43) to Eq. (42) yields

$$\Big(\Gamma_{jk}(\boldsymbol{n})\frac{\partial^2}{\partial s^2}+\rho\delta_{jk}w^2\Big)\hat{G}_{qj}(s)=-\delta_{qk}\delta(s), \tag{44}$$

$$\Big(\Gamma_{jk}(\boldsymbol{n})\frac{\partial^2}{\partial s^2}+\rho\delta_{jk}w^2\Big)\hat{G}_{4j}(s)=-(L_{45}^2-L_{55}L_{44})^{-1}(L_{55}L_{4k}-L_{45}L_{5k})\delta(s), \tag{45}$$

$$\Big(\Gamma_{jk}(\boldsymbol{n})\frac{\partial^2}{\partial s^2}+\rho\delta_{jk}w^2\Big)\hat{G}_{5j}(s)=-(L_{45}^2-L_{55}L_{44})^{-1}(L_{44}L_{5k}-L_{45}L_{4k})\delta(s), \tag{46}$$

in which

$$\Gamma_{jk}(\boldsymbol{n})=L_{jk}+(L_{45}L_{54}-L_{44}L_{55})^{-1}[L_{j4}L_{55}L_{4k}-L_{54}(L_{j5}L_{4k}+L_{j4}L_{5k})+L_{j5}L_{44}L_{5k}]. \tag{47}$$

Comparing the right side of Eqs. (45) and (46) with Eqs. (44) and (43) and noticing that $L_{JK}=L_{KJ}$, we have

$$\hat{G}_{JQ}=\hat{G}_{QJ}. \tag{48}$$

Therefore, Eqs. (45) and (46) will not be mentioned in the later discussion. We will solve $\hat{G}_{4j}(s)$ and $\hat{G}_{5j}(s)$ by Eq. (43) instead.

It is shown from Eq. (47) that the matrix $\boldsymbol{\Gamma}=[\Gamma_{jk}(\boldsymbol{n})]$ is symmetric and positive. By transforming the coordinates to the bases of the eigenspaces of $\boldsymbol{\Gamma}$, Eq. (44) can be reduced to a system of uncoupled 1-D Helmholtz equations. The eigenfunctions are given by

$$\Gamma_{jk}E_{km}=\lambda_m E_{jm},\quad m=1,2,3, \tag{49}$$

where λ_m is the eigenvalue corresponding to the eigenvector $\boldsymbol{E}_m=[E_{1m},E_{2m},E_{3m}]^{\mathrm{T}}$ of $\boldsymbol{\Gamma}$. It is worth mentioning that the summation convention does not and will not apply to the suffix m. It is easily proved that both the eigenvalues and eigenvectors are real. And here we take the eigenvectors as orthonormal bases, which gives

$$\boldsymbol{E}_m\cdot\boldsymbol{E}_n=\delta_{mn}. \tag{50}$$

The transformation of Eq. (44) is given by

$$\Big(\lambda_m\frac{\partial^2}{\partial s^2}+\rho w^2\Big)\tilde{G}_{qm}(s)=-E_{qm}\delta(s), \tag{51}$$

where

$$\tilde{G}_{qm}(s)=E_{jm}\hat{G}_{qj}, \tag{52}$$

and

$$\hat{G}_{qj}(s)=E_{nj}\tilde{G}_{qn}. \tag{53}$$

The solution of Eq. (51) was first obtained by Wang and Achenbach[11]:

$$\tilde{G}_{qm}=\frac{iE_{qm}}{2\rho c_m^2 k_m}\mathrm{e}^{\mathrm{i}k_m|s|}, \tag{54}$$

where the phase velocity c_m and the wave number k_m are defined respectively by

$$c_m=\sqrt{\lambda_m/\rho},\quad k_m=w/c_m. \tag{55}$$

From Eq. (53), we have

$$\hat{G}_{qj} = \sum_{m=1}^{3} \frac{iE_{qm}E_{jm}}{2\rho c_m^2 k_m} \mathrm{e}^{\mathrm{i}k_m|s|}. \tag{56}$$

The inverse transform of (54) is obtained by substituting Eq. (56) into Eq. (A2),

$$G_{qj}(\boldsymbol{x}) = \frac{1}{4\pi^2} \sum_{m=1}^{3} \oint_{|\boldsymbol{n}|=1} \frac{E_{qm}E_{jm}}{2\rho c_m^2} \int_{-\infty}^{\infty} \mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau) \frac{\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau} d\tau dl(\boldsymbol{n}). \tag{57a}$$

The same process of the derivation of G_{qj} is repeated to obtain G_{q4}, G_{q5}, G_{44}, G_{45} and G_{55}, which gives

$$G_{q4}(\boldsymbol{x}) = \frac{1}{4\pi^2} \sum_{m=1}^{3} \oint_{|\boldsymbol{n}|=1} \frac{E_{qm}E_{jm}(L_{54}^{-1}L_{j4} - L_{55}^{-1}L_{j5})}{2\rho c_m^2 (L_{55}^{-1}L_{45} - L_{54}^{-1}L_{44})} \int_{-\infty}^{\infty} \mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau) \frac{\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau} d\tau dl(\boldsymbol{n}), \tag{57b}$$

$$G_{q5}(\boldsymbol{x}) = \frac{1}{4\pi^2} \sum_{m=1}^{3} \oint_{|\boldsymbol{n}|=1} \frac{E_{qm}E_{jm}(L_{44}^{-1}L_{j4} - L_{45}^{-1}L_{j5})}{2\rho c_m^2 (L_{45}^{-1}L_{55} - L_{44}^{-1}L_{54})} \int_{-\infty}^{\infty} \mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau) \frac{\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau} d\tau dl(\boldsymbol{n}), \tag{57c}$$

$$\begin{aligned} G_{44}(\boldsymbol{x}) = \frac{1}{4\pi^2}\Big\{ \sum_{m=1}^{3} \oint_{|\boldsymbol{n}|=1} \Big[\frac{(L_{54}^{-1}L_{4q} - L_{55}^{-1}L_{5q})E_{qm}E_{jm}(L_{54}^{-1}L_{j4} - L_{55}^{-1}L_{j5})}{2\rho c_m^2 (L_{55}^{-1}L_{45} - L_{54}^{-1}L_{44})^2} \int_{-\infty}^{\infty} \mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau) \\ \cdot \frac{\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau} d\tau \Big] dl(\boldsymbol{n}) - \oint_{|\boldsymbol{n}|=1} \int_{-\infty}^{\infty} \frac{\mathrm{sgn}(\tau+\boldsymbol{n}\cdot\boldsymbol{x})}{2\tau(L_{55}^{-1}L_{45} - L_{54}^{-1}L_{44})L_{54}} d\tau dl(\boldsymbol{n}) \Big\}, \end{aligned} \tag{57d}$$

$$\begin{aligned} G_{45}(\boldsymbol{x}) = \frac{1}{4\pi^2}\Big\{ \sum_{m=1}^{3} \oint_{|\boldsymbol{n}|=1} \Big[\frac{(L_{44}^{-1}L_{4q} - L_{45}^{-1}L_{5q})E_{qm}E_{jm}(L_{54}^{-1}L_{j4} - L_{55}^{-1}L_{j5})}{2\rho c_m^2 (L_{55}^{-1}L_{45} - L_{54}^{-1}L_{44})(L_{45}^{-1}L_{55} - L_{44}^{-1}L_{54})} \int_{-\infty}^{\infty} \mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau) \\ \cdot \frac{\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau} d\tau \Big] dl(\boldsymbol{n}) - \oint_{|\boldsymbol{n}|=1} \int_{-\infty}^{\infty} \frac{\mathrm{sgn}(\tau+\boldsymbol{n}\cdot\boldsymbol{x})}{2\tau(L_{45}^{-1}L_{55} - L_{44}^{-1}L_{54})L_{44}} d\tau dl(\boldsymbol{n}) \Big\}, \end{aligned} \tag{57e}$$

$$\begin{aligned} G_{55}(\boldsymbol{x}) = \frac{1}{4\pi^2}\Big\{ \sum_{m=1}^{3} \oint_{|\boldsymbol{n}|=1} \Big[\frac{(L_{44}^{-1}L_{4q} - L_{45}^{-1}L_{5q})E_{qm}E_{jm}(L_{44}^{-1}L_{j4} - L_{45}^{-1}L_{j5})}{2\rho c_m^2 (L_{45}^{-1}L_{55} - L_{44}^{-1}L_{54})^2} \int_{-\infty}^{\infty} \mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau) \\ \cdot \frac{\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau} d\tau \Big] dl(\boldsymbol{n}) - \oint_{|\boldsymbol{n}|=1} \int_{-\infty}^{\infty} \frac{\mathrm{sgn}(\tau+\boldsymbol{n}\cdot\boldsymbol{x})}{2\tau(L_{44}^{-1}L_{54} - L_{45}^{-1}L_{55})L_{45}} d\tau dl(\boldsymbol{n}) \Big\}. \end{aligned} \tag{57f}$$

Equations (57a)–(57f) are inserted into Eqs. (25), (26), (29) and (30) to get the corresponding operators,

$$\begin{aligned} N_{ijq}(\boldsymbol{x},t) &= -\frac{1}{4\pi^2} \sum_{m=1}^{3} \oint_{|\boldsymbol{n}|=1} \frac{E_{qm}E_{jm}n_i}{2\rho c_m^2} \int_{-\infty}^{\infty} \Big(\frac{2\delta(\boldsymbol{n}\cdot\boldsymbol{x}+\tau)}{\tau} + \frac{ik_m \mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau} \Big) d\tau dl(\boldsymbol{n}) \\ &= -\frac{1}{4\pi^2} \sum_{m=1}^{3} \oint_{|\boldsymbol{n}|=1} \frac{E_{qm}E_{jm}n_i}{2\rho c_m^2} \Big(\int_{-\infty}^{\infty} \frac{ik_m \mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau} d\tau - \frac{2}{\boldsymbol{n}\cdot\boldsymbol{x}} \Big) dl(\boldsymbol{n}), \end{aligned} \tag{58a}$$

$$N_{i4q}(\boldsymbol{x},t) = -\frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{jm}n_i(L_{54}^{-1}L_{j4}-L_{55}^{-1}L_{j5})}{2\rho c_m^2(L_{55}^{-1}L_{45}-L_{54}^{-1}L_{44})} \cdot \Big(\int_{-\infty}^{\infty}\frac{ik_m \mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau}d\tau - \frac{2}{\boldsymbol{n}\cdot\boldsymbol{x}}\Big)dl(\boldsymbol{n}), \tag{58b}$$

$$N_{i44}(\boldsymbol{x},t) = -\frac{1}{4\pi^2}\Big\{\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\Big[\frac{(L_{54}^{-1}L_{4q}-L_{55}^{-1}L_{5q})E_{qm}E_{jm}n_i(L_{54}^{-1}L_{j4}-L_{55}^{-1}L_{j5})}{2\rho c_m^2(L_{55}^{-1}L_{45}-L_{54}^{-1}L_{44})^2} \cdot \Big(-\frac{2}{\boldsymbol{n}\cdot\boldsymbol{x}}+\int_{-\infty}^{\infty}\frac{ik_m \mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau}d\tau\Big)\Big]dl(\boldsymbol{n}) - \oint_{|\boldsymbol{n}|=1}\frac{n_i}{\boldsymbol{n}\cdot\boldsymbol{x}(L_{55}^{-1}L_{45}-L_{54}^{-1}L_{44})L_{54}}dl(\boldsymbol{n})\Big\}, \tag{58c}$$

$$N_{i54}(\boldsymbol{x},t) = -\frac{1}{4\pi^2}\Big\{\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\Big[\frac{(L_{44}^{-1}L_{4q}-L_{45}^{-1}L_{5q})E_{qm}E_{jm}n_i(L_{54}^{-1}L_{j4}-L_{55}^{-1}L_{j5})}{2\rho c_m^2(L_{55}^{-1}L_{45}-L_{54}^{-1}L_{44})(L_{45}^{-1}L_{55}-L_{44}^{-1}L_{54})} \cdot \Big(-\frac{2}{\boldsymbol{n}\cdot\boldsymbol{x}}+\int_{-\infty}^{\infty}\frac{ik_m \mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau}d\tau\Big)\Big]dl(\boldsymbol{n}) - \oint_{|\boldsymbol{n}|=1}\frac{n_i}{\boldsymbol{n}\cdot\boldsymbol{x}(L_{45}^{-1}L_{55}-L_{44}^{-1}L_{54})L_{44}}dl(\boldsymbol{n})\Big\}, \tag{58d}$$

$$(N_x)_{ijql} = \frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{jm}n_in_l}{2\rho c_m^2}\Big[\int_{-\infty}^{\infty}\frac{k_m^2\mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau)\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau}d\tau - \frac{2}{(\boldsymbol{n}\cdot\boldsymbol{x})^2}\Big]dl(\boldsymbol{n}), \tag{59a}$$

$$(N_x)_{i4ql} = \frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{jm}n_in_l(L_{54}^{-1}L_{j4}-L_{55}^{-1}L_{j5})}{2\rho c_m^2(L_{55}^{-1}L_{45}-L_{54}^{-1}L_{44})} \cdot \Big[\int_{-\infty}^{\infty}\frac{k_m^2\mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau)\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau}d\tau - \frac{2}{(\boldsymbol{n}\cdot\boldsymbol{x})^2}\Big]dl(\boldsymbol{n}), \tag{59b}$$

$$(N_x)_{i44l} = \frac{1}{4\pi^2}\Big\{\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{(L_{54}^{-1}L_{4q}-L_{55}^{-1}L_{5q})E_{qm}E_{jm}n_in_j(L_{54}^{-1}L_{j4}-L_{55}^{-1}L_{j5})}{2\rho c_m^2(L_{55}^{-1}L_{45}-L_{54}^{-1}L_{44})^2} \cdot \Big[\int_{-\infty}^{\infty}\frac{k_m^2\mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau)\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau}d\tau - \frac{2}{(\boldsymbol{n}\cdot\boldsymbol{x})^2}\Big]dl(\boldsymbol{n}) - \oint_{|\boldsymbol{n}=1|}\frac{n_in_j}{(\boldsymbol{n}\cdot\boldsymbol{x})^2(L_{55}^{-1}L_{45}-L_{54}^{-1}L_{44})L_{54}}dl(\boldsymbol{n})\Big\}, \tag{59c}$$

$$(N_x)_{i54l} = \frac{1}{4\pi^2}\Big\{\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{(L_{44}^{-1}L_{4q}-L_{45}^{-1}L_{5q})E_{qm}E_{jm}n_in_l(L_{54}^{-1}L_{j4}-L_{55}^{-1}L_{j5})}{2\rho c_m^2(L_{55}^{-1}L_{45}-L_{54}^{-1}L_{44})(L_{45}^{-1}L_{55}-L_{44}^{-1}L_{54})} \cdot \Big[\int_{-\infty}^{\infty}\frac{k_m^2\mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau)\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau}d\tau - \frac{2}{(\boldsymbol{n}\cdot\boldsymbol{x})^2}\Big]dl(\boldsymbol{n}) - \oint_{|\boldsymbol{n}=1|}\frac{n_in_l}{(\boldsymbol{n}\cdot\boldsymbol{x})^2(L_{45}^{-1}L_{55}-L_{44}^{-1}L_{54})L_{44}}dl(\boldsymbol{n})\Big\}, \tag{59d}$$

and

$$M_{QJ}(\boldsymbol{x}) = -iwG_{QJ}(\boldsymbol{x}), \tag{59e}$$

$$(M_x)_{JQl} = iwN_{JQl}, \tag{59f}$$

$$(N_t)_{iJQ} = -iwN_{iJQ}, \tag{59g}$$

$$(M_t)_{JQ} = -iwM_{JQ}. \tag{59h}$$

It should be noted that in Eqs. (58a)–(59h), $i = 1, 2$ and $l = 1, 2$ for a two-dimensional problem, which distinct themselves from other lowercase subscripts that take values from 1 to 3. This rule also applies to later discussion. The missing equations in Eqs. (58a)–(59h) can be obtained by switching corresponding subscripts. The lengthy expressions in Eqs. (58a)–(59h) can be treated numerically. Take Eq. (59d) as an example: for the first term with kernel $\int_{-\infty}^{\infty}\{[k_m^2 \mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau)\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}]/\tau\}d\tau$, we can separate the integration into two parts to eliminate $\mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau)$, and then we get expressions with kernel $\int_{-\infty}^{-\boldsymbol{n}\cdot\boldsymbol{x}}(\mathrm{e}^{\mathrm{i}k_m\tau}/\tau)d\tau$, which can be solved by series expansion. The second and third terms of Eq. (59d) will be solved in Section 5. Combination of Eqs. (58a)–(59h) and Eqs. (27)–(28) yields the generalized displacement described by Eq. (22). From Eq. (22), we learn that the total displacement field is composed of the incident field $\boldsymbol{u}_0$ and the scattered field $\boldsymbol{v}$, which is given by

$$\boldsymbol{v} = -\boldsymbol{N}\boldsymbol{\tau} - \boldsymbol{M}\boldsymbol{\pi}. \tag{60}$$

For piezoelectric materials, Eqs. (57a)–(59h) reduce to

$$G_{qj}(\boldsymbol{x}) = \frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{jm}}{2\rho c_m^2}\int_{-\infty}^{\infty}\mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau)\frac{\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau}d\tau dl(\boldsymbol{n}), \tag{61a}$$

$$G_{q4}(\boldsymbol{x}) = -\frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{jm}L_{j4}}{2\rho c_m^2 L_{44}}\int_{-\infty}^{\infty}\mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau)\frac{\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau}d\tau dl(\boldsymbol{n}), \tag{61b}$$

$$\begin{aligned} G_{44}(\boldsymbol{x}) = \frac{1}{4\pi^2}\Big\{&\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\Big[\frac{L_{4q}E_{qm}E_{jm}L_{j4}}{2\rho c_m^2 L_{44}^2}\int_{-\infty}^{\infty}\mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau)\frac{\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau}d\tau\Big]dl(\boldsymbol{n}) \\ &-\oint_{|\boldsymbol{n}|=1}\int_{-\infty}^{\infty}\frac{\mathrm{sgn}(\tau+\boldsymbol{n}\cdot\boldsymbol{x})}{2\tau L_{44}}d\tau dl(\boldsymbol{n})\Big\}, \end{aligned} \tag{61c}$$

$$N_{ijq}(\boldsymbol{x},t) = -\frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{jm}n_i}{2\rho c_m^2}\Big(\int_{-\infty}^{\infty}\frac{ik_m\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau}d\tau - \frac{2}{\boldsymbol{n}\cdot\boldsymbol{x}}\Big)dl(\boldsymbol{n}), \tag{61d}$$

$$N_{i4q}(\boldsymbol{x},t) = \frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{jm}n_iL_{j4}}{2\rho c_m^2 L_{44}}\Big(\int_{-\infty}^{\infty}\frac{ik_m\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau}d\tau - \frac{2}{\boldsymbol{n}\cdot\boldsymbol{x}}\Big)dl(\boldsymbol{n}), \tag{61e}$$

$$\begin{aligned} N_{i44}(\boldsymbol{x},t) = -\frac{1}{4\pi^2}\Big\{&\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\Big[\frac{L_{4q}E_{qm}E_{jm}n_iL_{j4}}{2\rho c_m^2 L_{44}^2}\Big(-\frac{2}{\boldsymbol{n}\cdot\boldsymbol{x}}+\int_{-\infty}^{\infty}\frac{ik_m\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau}d\tau\Big)\Big]dl(\boldsymbol{n}) \\ &+\oint_{|\boldsymbol{n}|=1}\frac{n_i}{L_{44}\boldsymbol{n}\cdot\boldsymbol{x}}dl(\boldsymbol{n})\Big\}, \end{aligned} \tag{61f}$$

and

$$(N_x)_{ijql} = \frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{jm}n_in_l}{2\rho c_m^2}\Big[\int_{-\infty}^{\infty}\frac{k_m^2\mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau)\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau}d\tau - \frac{2}{(\boldsymbol{n}\cdot\boldsymbol{x})^2}\Big]dl(\boldsymbol{n}), \tag{62a}$$

$$(N_x)_{i4ql} = -\frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{jm}n_in_lL_{j4}}{2\rho c_m^2L_{44}}\Big[\int_{-\infty}^{\infty}\frac{k_m^2\mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau)\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau}d\tau - \frac{2}{(\boldsymbol{n}\cdot\boldsymbol{x})^2}\Big]dl(\boldsymbol{n}), \tag{62b}$$

$$(N_x)_{i44l} = \frac{1}{4\pi^2}\Big\{\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{L_{4q}E_{qm}E_{jm}n_in_jL_{j4}}{2\rho c_m^2L_{44}}\Big[\int_{-\infty}^{\infty}\frac{k_m^2\mathrm{sgn}(\boldsymbol{n}\cdot\boldsymbol{x}+\tau)\mathrm{e}^{\mathrm{i}k_m|\boldsymbol{n}\cdot\boldsymbol{x}+\tau|}}{\tau}d\tau - \frac{2}{(\boldsymbol{n}\cdot\boldsymbol{x})^2}\Big]dl(\boldsymbol{n}) + \oint_{|\boldsymbol{n}|=1}\frac{n_in_j}{(\boldsymbol{n}\cdot\boldsymbol{x})^2L_{44}}dl(\boldsymbol{n})\Big\}. \tag{62c}$$

For the reduced material, Eq. (47) becomes

$$\Gamma_{jk}(\boldsymbol{n}) = L_{jk} + L_{44}^{-1}L_{j4}L_{4k}. \tag{63}$$

It should be noticed that for the reduced material, terms corresponding to the magnetic properties in Eq. (2) are omitted. It is obvious that if we exchange the related terms, Eqs. (61a)–(61f) can also apply to piezomagnetic materials.

4 Scattering cross-section

The scattering cross-section κ of inclusion is defined as the ratio of the total mean rate of energy outflow corresponding to the scattered field $\boldsymbol{v}$ to the mean energy flow of the incident wave in direction $\boldsymbol{n}^0$. The mean energy flux of $\boldsymbol{v}$ has components

$$Y_i = -\frac{1}{4}iw(\sigma_{iJ}\bar{v}_J - \bar{\sigma}_{iJ}v_J), \tag{64}$$

where σ_{iJ} is the generalized stress field associated with $\boldsymbol{v}$, and the superposed bar denotes complex conjugation. The mean rate of energy radiation out of a two-dimensional domain is then obtained as

$$E = \int_{\partial S} Y_i n_i^0 dl. \tag{65}$$

Using Gauss' theorem, we get

$$E = -\frac{1}{4}iw\int_S(\sigma_{iJ,i}\bar{v}_J - \bar{\sigma}_{iJ,i}v_J + \sigma_{iJ}\bar{v}_{J,i} - \bar{\sigma}_{iJ}v_{J,i})dS, \tag{66}$$

which can be transformed into

$$E = -\frac{1}{4}iw\int_S(\tau_{iJ}\bar{v}_{J,i} - \bar{\tau}_{iJ}v_{J,i})dS - \frac{1}{4}w^2\int_S(\pi_J\bar{v}_J + \bar{\pi}_Jv_J)dS. \tag{67}$$

In the above equation, $\boldsymbol{\tau}$ and $\boldsymbol{\pi}$ are non-zero only over the plane S occupied by the inclusion. The mean energy flux of the incident wave can be obtained from Eq. (32) as

$$E^0 = \frac{\rho w^3}{2k_0}a_ja_j, \tag{68}$$

where a_j are the components of the polarization vector $\boldsymbol{a}$ given by Eq. (31). Finally, we obtain the expression of the scattering cross-section,

$$\kappa = E/E^0. \tag{69}$$

5 Analytical results under certain simplification

Equations (27) and (28) can be simplified considerably in the low frequency range, or the so-called Rayleigh limit. Moreover, if the diameter of the inclusion is much smaller than the wavelength of the incident wave, $\boldsymbol{\tau}$ and $\boldsymbol{\pi}$ can be considered as constants over the inclusion. Retention of the lowest terms reduces the equations to

$$(\Delta C)^{-1}_{iJQl}\tau_{iJ} + \int_S (N_x^\infty)_{iJQl} d\boldsymbol{x}'\tau_{iJ} = -ik_0 a_Q n_l^0, \tag{70}$$

$$(\Delta\rho)^{-1}\pi_j = -iwa_j. \tag{71}$$

The integral in Eq. (70) is the static limit of operator N_x, which can be treated as follows:

$$\int_S (N_x^\infty)_{iJQl} d\boldsymbol{x}' = \frac{d[\int_S N_{iJQ}^\infty(\boldsymbol{x}-\boldsymbol{x}')d\boldsymbol{x}']}{dx_l}, \tag{72}$$

where N_{iJQ}^∞ can be obtained from Eqs. (58a)–(58f) by merely retaining the terms with $\frac{1}{\boldsymbol{n}\cdot\boldsymbol{x}}$. Equation (72) can be solved explicitly, whose result is a constant tensor given in Appendix B. In Eqs. (70)–(71), $\boldsymbol{\tau}$ and $\boldsymbol{\pi}$ are directly solved as

$$\tau_{iJ} = -ik_0[(\Delta C)^{-1} + \int_S (N_x^\infty) d\boldsymbol{x}']^{-1}_{iJQl} a_Q n_l^0, \tag{73}$$

$$\pi_j = -i(\Delta\rho)wa_j. \tag{74}$$

The scattering cross-section is evaluated for this simplified case. From Eqs. (67)–(69), we get

$$\kappa = \frac{(\pi ab)^2 w^3}{2E^0}[\tau_{iJ}(\Delta N_x)_{iJQl}\bar{\tau}_{lQ} + \pi_k \Delta M_{kp}\bar{\pi}_p], \tag{75}$$

where

$$(\Delta N_x)_{ijql} = \frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{mj}n_i n_l}{\rho c_m^5}\Big[\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+1)(2i+1)!} - \frac{1}{2}\Big]dl(\boldsymbol{n}), \tag{76a}$$

$$(\Delta N_x)_{i4ql} = \frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{mj}n_i n_l(L_{54}^{-1}L_{j4} - L_{55}^{-1}L_{j5})}{\rho c_m^5(L_{55}^{-1}L_{45} - L_{54}^{-1}L_{44})} \cdot\Big[\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+1)(2i+1)!} - \frac{1}{2}\Big]dl(\boldsymbol{n}), \tag{76b}$$

$$(\Delta N_x)_{i5ql} = \frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{mj}n_i n_l(L_{44}^{-1}L_{j4} - L_{45}^{-1}L_{j5})}{\rho c_m^5(L_{45}^{-1}L_{55} - L_{44}^{-1}L_{54})} \cdot\Big[\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+1)(2i+1)!} - \frac{1}{2}\Big]dl(\boldsymbol{n}), \tag{76c}$$

$$(\Delta N_x)_{i44l} = \frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{(L_{54}^{-1}L_{4q} - L_{55}^{-1}L_{5q})E_{qm}E_{mj}n_i n_j(L_{54}^{-1}L_{j4} - L_{55}^{-1}L_{j5})}{\rho c_m^5(L_{55}^{-1}L_{45} - L_{54}^{-1}L_{44})^2} \cdot\Big[\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+1)(2i+1)!} - \frac{1}{2}\Big]dl(\boldsymbol{n}), \tag{76d}$$

$$(\Delta N_x)_{i54l} = \frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{(L_{44}^{-1}L_{4q}-L_{45}^{-1}L_{5q})E_{qm}E_{mj}n_in_l(L_{54}^{-1}L_{j4}-L_{55}^{-1}L_{j5})}{\rho c_m^5(L_{55}^{-1}L_{45}-L_{54}^{-1}L_{44})(L_{45}^{-1}L_{55}-L_{44}^{-1}L_{54})} \cdot\Big[\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+1)(2i+1)!}-\frac{1}{2}\Big]dl(\boldsymbol{n}), \tag{76e}$$

$$(\Delta N_x)_{i55l} = \frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{(L_{54}^{-1}L_{4q}-L_{55}^{-1}L_{5q})E_{qm}E_{mj}n_in_j(L_{54}^{-1}L_{j4}-L_{55}^{-1}L_{j5})}{2\rho c_m^5(L_{55}^{-1}L_{45}-L_{54}^{-1}L_{44})^2} \cdot\Big[\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+1)(2i+1)!}-\frac{1}{2}\Big]dl(\boldsymbol{n}), \tag{76f}$$

$$\Delta M_{kp} = \frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{mj}}{\rho c_m^3}\Big[\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+1)(2i+1)!}-\frac{1}{2}\Big]dl(\boldsymbol{n}). \tag{76g}$$

In the derivation of Eq. (75), the parities of Eqs. (58a)–(59j) are used. The detailed deduction of Eqs. (76a)–(76g) are given in Appendix C.

For general piezoelectric materials, Eqs. (76a)–(76g) reduce to

$$(\Delta N_x)_{ijql} = \frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{mj}n_in_l}{\rho c_m^5}\Big[\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+1)(2i+1)!}-\frac{1}{2}\Big]dl(\boldsymbol{n}), \tag{77a}$$

$$(\Delta N_x)_{i4ql} = -\frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{mj}n_in_lL_{j4}}{\rho c_m^5L_{44}}\Big[\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+1)(2i+1)!}-\frac{1}{2}\Big]dl(\boldsymbol{n}), \tag{77b}$$

$$(\Delta N_x)_{i44l} = \frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{L_{4q}E_{qm}E_{mj}n_in_jL_{j4}}{\rho c_m^5L_{44}^2}\Big[\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+1)(2i+1)!}-\frac{1}{2}\Big]dl(\boldsymbol{n}), \tag{77c}$$

$$\Delta M_{jp} = \frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{mj}}{\rho c_m^3}\Big[\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+1)(2i+1)!}-\frac{1}{2}\Big]dl(\boldsymbol{n}). \tag{77d}$$

6 Numerical examples

In this section, the scattering cross-section is calculated for the piezoelectric composite, which consists of a single inclusion (with two types of different material constants: $BaTiO_3$ and $BaTiO_3$ rigidity) and a PZT-5H-matrix. The matrix and the inclusion are transversely isotropic piezoelectric material with the symmetry axis x_2, and the two-dimensional problem is calculated in the x_1-x_2 plane. (Notice that the symmetry axis is set in the x_2-direction. So the material is not symmetric in x_1- and x_2-directions.) The non-zero elements of material constants are the $BaTiO_3$-inclusion:

$$\begin{cases} C_{11}^* = 166\ \text{GPa}, \quad C_{22}^* = 162\ \text{GPa}, \quad C_{12}^* = 78\ \text{GPa}, \quad C_{13}^* = 77\ \text{GPa}, \quad C_{44}^* = 43\ \text{GPa}, \\ e_{21}^* = -4.4\ \text{C}\cdot\text{m}^{-2}, \quad e_{22}^* = 18.6\ \text{C}\cdot\text{m}^{-2}, \quad e_{15}^* = 11.6\ \text{C}\cdot\text{m}^{-2}, \\ \varepsilon_{11}^* = 11.2\times10^{-9}\ \text{C}\cdot\text{N}^{-1}\cdot\text{m}^{-2}, \quad \varepsilon_{22}^* = 12.6\times10^{-9}\ \text{C}\cdot\text{N}^{-1}\cdot\text{m}^{-2}, \quad \rho^* = 5\,700\ \text{kg}\cdot\text{m}^{-3}; \end{cases} \tag{78}$$

and the PZT-5H-matrix:

$$\begin{cases} C_{11} = 126\ \text{GPa}, \quad C_{22} = 117\ \text{GPa}, \quad C_{12} = 53\ \text{GPa}, \quad C_{13} = 55\ \text{GPa}, \quad C_{44} = 35.5\ \text{GPa}, \\ e_{21} = -6.5\ \text{C}\cdot\text{m}^{-2}, \quad e_{22} = 23.3\ \text{C}\cdot\text{m}^{-2}, \quad e_{15} = 17.0\ \text{C}\cdot\text{m}^{-2}, \\ \varepsilon_{11} = 15.1\times10^{-9}\ \text{C}\cdot\text{N}^{-1}\cdot\text{m}^{-2}, \quad \varepsilon_{22} = 13.0\times10^{-9}\ \text{C}\cdot\text{N}^{-1}\cdot\text{m}^{-2}, \quad \rho = 7\,500\ \text{kg}\cdot\text{m}^{-3}. \end{cases} \tag{79}$$

And for the $BaTiO_3$ rigidity inclusion, the constants of the material are set to be infinite, but the density of the material remains the same.

The generalized stress-strain relationship in this case is given by

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \\ D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 & 0 & e_{21} \\ C_{12} & C_{22} & 0 & 0 & 0 & 0 & e_{22} \\ 0 & 0 & C_{44} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{13}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & e_{15} & 0 \\ 0 & 0 & 0 & 0 & e_{15} & -\varepsilon_{11} & 0 \\ e_{21} & e_{22} & 0 & 0 & 0 & 0 & -\varepsilon_{22} \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \\ -E_1 \\ -E_2 \end{bmatrix}. \tag{80}$$

We consider the following problem: where the incident wave propagating in the x_1-direction in matrix, with the magnitude of displacement $\boldsymbol{a} = [1\ 0\ 0]^{\mathrm{T}}$, collides on the elliptical inclusion. The size of the inclusion is illustrated in Fig. 1. For two different types of materials of the inclusion, we calculate the scattering cross-section as a function of ε. As ε increases, the volume of the inclusion also increases linearly. The results are illustrated in Fig. 2. All results are normalized with respect to w^4a^4, where a is the length shown in Fig. 1, and w is given by Eq. (31).

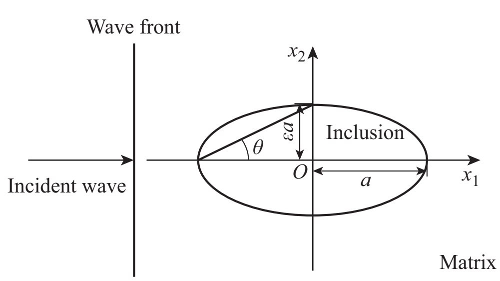

Fig. 1 Illustration of the scattering problem considered

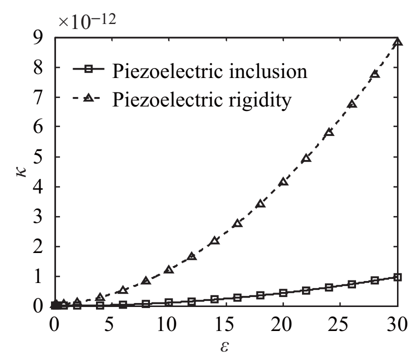

Fig. 2 Scattering cross-section κ as a function of ε

From the upper result, we see that in both cases ($BaTiO_3$ inclusion and $BaTiO_3$ rigidity inclusion), the size effect predominates. Because the scattering cross-section is an integration over the area occupied by the whole inclusion, when the size of the inclusion increases with ε, the value of the scattering cross-section will also increase. However, because the dependence of the values of the scattering cross-section on the size of the inclusion is too strong, it actually conceals the effect of the shape of the inclusion. To see this clearly, we make another calculation of the scattering cross-section change as a function of θ, which is shown in Fig. 1. It is worth mentioning that in this calculation, the size of the inclusion is kept invariant as θ changes. To compare piezoelectric materials with normal elastic materials, we first calculate the scattering cross-section for matrices and the inclusion without piezoelectricity, yet having the same elastic properties with the PZT-5H matrices and the $BaTiO_3$ inclusion (we call them the elastic $BaTiO_3$ and the elastic PZT-5H), whose nonzero material constants are given by the elastic $BaTiO_3$-inclusion:

$$\begin{cases} C_{11}^* = 166\ \text{GPa}, \quad C_{22}^* = 162\ \text{GPa}, \quad C_{12}^* = 78\ \text{GPa}, \quad C_{13}^* = 77\ \text{GPa}, \quad C_{44}^* = 43\ \text{GPa}, \\ e_{21}^* = 0\ \text{C}\cdot\text{m}^{-2}, \quad e_{22}^* = 0\ \text{C}\cdot\text{m}^{-2}, \quad e_{15}^* = 0\ \text{C}\cdot\text{m}^{-2}, \quad \varepsilon_{11}^* = 0\ \text{C}\cdot\text{N}^{-2}\cdot\text{m}^{-2}, \\ \varepsilon_{22}^* = 0\ \text{C}\cdot\text{N}^{-2}\cdot\text{m}^{-2}, \quad \rho^* = 5\,700\ \text{kg}\cdot\text{m}^{-3}; \end{cases}$$

and the elastic PZT-5H-matrix:

$$\begin{cases} C_{11} = 126\ \text{GPa}, \quad C_{22} = 117\ \text{GPa}, \quad C_{12} = 53\ \text{GPa}, \quad C_{13} = 55\ \text{GPa}, \quad C_{44} = 35.5\ \text{GPa}, \\ e_{21} = 0\ \text{C}\cdot\text{m}^{-2}, \quad e_{22} = 0\ \text{C}\cdot\text{m}^{-2}, \quad e_{15} = 0\ \text{C}\cdot\text{m}^{-2}, \quad \varepsilon_{11} = 0\ \text{C}\cdot\text{N}^{-2}\cdot\text{m}^{-2}, \\ \varepsilon_{22} = 0\ \text{C}\cdot\text{N}^{-2}\cdot\text{m}^{-2}, \quad \rho = 7\,500\ \text{kg}\cdot\text{m}^{-3}. \end{cases}$$

The $BaTiO_3$ rigidity inclusion is also considered in this case. The results are illustrated in Figs. 3 and 4.

The results for the piezoelectric matrices and inclusion are illustrated in Figs. 5 and 6, with parameters in Eqs. (78) and (79).

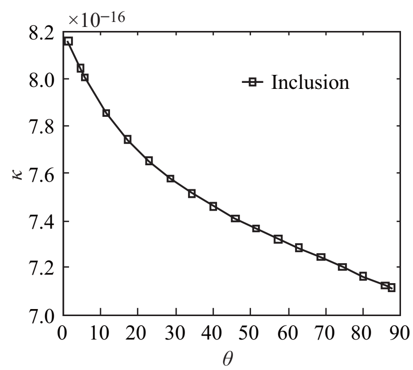

Fig. 3 Scattering cross-section κ as a function of θ, the materials of matrices and inclusion in this case are elastic PZT-5H and elastic $BaTiO_3$, respectively

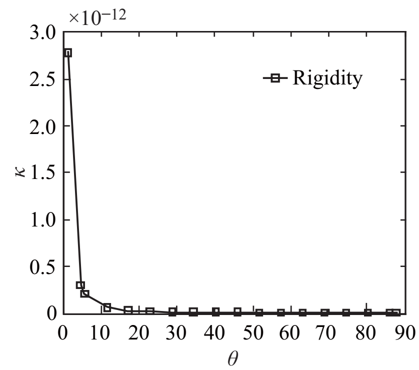

Fig. 4 Scattering cross-section κ as a function of θ, the materials of matrices and inclusion in this case are elastic PZT-5H and elastic $BaTiO_3$ rigidity, respectively

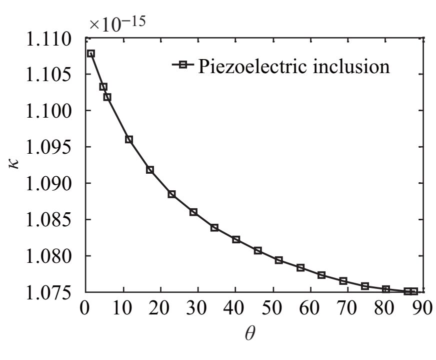

Fig. 5 Scattering cross-section κ as a function of θ, the materials of matrices and inclusion in this case are PZT-5H and $BaTiO_3$, respectively

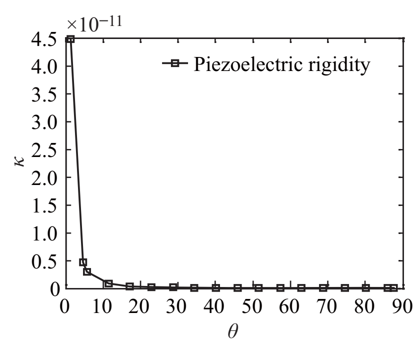

Fig. 6 Scattering cross-section κ as a function of θ, the materials of matrices and inclusion in this case are PZT-5H and $BaTiO_3$ rigidity, respectively

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Appendix A

Consider function $f(\boldsymbol{x})$ defined in R^2. The Radon transform of $f(\boldsymbol{x})$ is defined as

$$\hat{f}(s,\boldsymbol{n}) = R[f(\boldsymbol{x})] = \int f(\boldsymbol{x})\delta(s-\boldsymbol{n}\cdot\boldsymbol{x})d\boldsymbol{x}, \tag{A1}$$

where $\boldsymbol{n}$ is a unite vector, and δ is the two-dimensional Dirac delta. The Radon transform integral $f(\boldsymbol{x})$ over all curves is defined by $\boldsymbol{n}\cdot\boldsymbol{x} = s$. The inverse Radon transform is defined as

$$f(\boldsymbol{x}) = -\frac{\mathrm{i}}{4\pi}\oint_{|\boldsymbol{n}|=1} H_s\left[\frac{d}{ds}\hat{f}(s,\boldsymbol{n})|_{s=\boldsymbol{n}\cdot\boldsymbol{x}}\right]dl(\boldsymbol{n}), \tag{A2}$$

where H_s is the Hilbert transform with respect to s defined as

$$H_s[f(s,\boldsymbol{n})] = \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\frac{f(\tau,\boldsymbol{n})}{s-\tau}d\tau. \tag{A3}$$

Helgason[12] provided detailed description of the properties of the Radon transform.

Appendix B

Using the techniques performed by Li and Wang[13], Eqs. (72)–(74) are solved as below:

$$N_{ijq}^{\infty} = \frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{mj}n_i}{\rho c_m^2(\boldsymbol{n}\cdot\boldsymbol{x})}dl(\boldsymbol{n}) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\frac{A_{qj}(\boldsymbol{e}_1+\zeta\boldsymbol{e}_2)\zeta_i}{D(\boldsymbol{e}_1+\zeta\boldsymbol{e}_2)(x_1+\zeta x_2)}d\zeta, \tag{B1}$$

where $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$ and $[\zeta_1, \zeta_2] = [1, \zeta]$. The integration of Eq. (B1) with respect to $d\boldsymbol{x}'$ has been done by Eshelby[8], which is proved to be linearly dependent on $\boldsymbol{x}$. Therefore, we arrive at the conclusion that $\frac{d[\int_S N^{\infty}_{ijq}(\boldsymbol{x}-\boldsymbol{x}')d\boldsymbol{x}']}{dx_l}$ are constants, which are given by

$$\int_S (N^{\infty}_x)_{ijql} d\boldsymbol{x}' = \frac{1}{\pi\varepsilon}\int_{-\infty}^{\infty} \frac{A_{qj}(\boldsymbol{e}_1 + \zeta\boldsymbol{e}_2)\zeta_{il}}{D(\boldsymbol{e}_1 + \zeta\boldsymbol{e}_2)(1/\varepsilon^2 + \zeta^2)} d\zeta, \tag{B2}$$

where $[\zeta_{il}] = \begin{bmatrix} 1 & \zeta \\ \zeta & \zeta^2 \end{bmatrix}^{\mathrm{T}}$ and $\varepsilon = b/a$, with a and b denoting the axis lengths of the elliptic inclusion in the directions of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, respectively. In the derivation of Eq. (B2), some complicated terms are omitted for that according to Eq. (67), only imaginary parts of $\tau_{iJ}\bar{v}_{J,i}$ contribute to the scattering section. For the most general cases where the roots for $D(\boldsymbol{e}_1 + \zeta\boldsymbol{e}_2) = 0$ are all distinct, Eq. (B2) can be solved explicitly as

$$\int_S (N^{\infty}_x)_{ijql} d\boldsymbol{x}' = -\frac{2}{\varepsilon}\mathrm{Im}\left[\sum_{m=1}^{6} \frac{A_{qj}(\boldsymbol{\zeta}_m)(\zeta_{il})_m}{a_6(\zeta_m - \zeta_m^*)\prod\limits_{\substack{k=1\\k\neq m}}^{6}(\zeta_m - \zeta_k^*)(\zeta_m - \zeta_k)}\right], \tag{B3}$$

where a_6 is the coefficient of the term ζ^{10} in $D(\boldsymbol{e}_1 + \zeta\boldsymbol{e}_2)$, ζ_m $(m = 1, 2, \cdots, 5)$ are the roots of $D(\boldsymbol{e}_1 + \zeta\boldsymbol{e}_2) = 0$ with positive imaginary parts, ζ_6 equals $\mathrm{i}(1/\varepsilon)$, and ζ_m^* is the conjugate of ζ_m. Other terms can be obtained in the same way as

$$\int_S (N^{\infty}_x)_{i4ql} d\boldsymbol{x}' = -\frac{2}{\varepsilon}\mathrm{Im}\left\{\sum_{m=1}^{8} \frac{A_{qj}(\zeta_m)(\zeta_{il})_m[L_{j4}(\zeta_m)L_{55}(\zeta_m) - L_{54}(\zeta_m)L_{j5}(\zeta_m)]}{a_6a_4(\zeta_m - \zeta_m^*)\prod\limits_{\substack{k=1\\k\neq m}}^{8}(\zeta_m - \zeta_k^*)(\zeta_m - \zeta_k)}\right\}, \tag{B4}$$

where a_4 is the coefficient of the term ζ^4 in $L_{45}L_{54} - L_{44}L_{55}$; ζ_7 and ζ_8 are roots with positive imaginary parts of the equation $(L_{45}L_{54} - L_{44}L_{55}) = 0$.

$$\int_S (N^{\infty}_x)_{i5ql} d\boldsymbol{x}' = -\frac{2}{\varepsilon}\mathrm{Im}\left\{\sum_{m=1}^{8} \frac{A_{qj}(\zeta_m)(\zeta_{il})_m[L_{45}(\zeta_m)L_{j4}(\zeta_m) - L_{44}(\zeta_m)L_{j5}(\zeta_m)]}{a_6(-a_4)(\zeta_m - \zeta_m^*)\prod\limits_{\substack{k=1\\k\neq m}}^{8}(\zeta_m - \zeta_k^*)(\zeta_m - \zeta_k)}\right\}. \tag{B5}$$

The other equations are of the similar form and too lengthy to be presented here.

Appendix C

From Eq. (67), we learn that only the imaginary and even terms in Eqs. (59a)–(59f) make contribution to the scattering cross-section. Therefore,

$$\begin{aligned}(\Delta N_x)_{ijql} = {} & \frac{1}{4w^3\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{jm}n_in_l}{2\rho c_m^2} \\ & \cdot\left[\int_{-\boldsymbol{n}\cdot\boldsymbol{x}}^{\infty}\frac{k_m^2\mathrm{e}^{\mathrm{i}k_m(\boldsymbol{n}\cdot\boldsymbol{x}+\tau)}}{\tau}d\tau - \int_{-\infty}^{-\boldsymbol{n}\cdot\boldsymbol{x}}\frac{k_m^2\mathrm{e}^{-\mathrm{i}k_m(\boldsymbol{n}\cdot\boldsymbol{x}+\tau)}}{\tau}d\tau\right]dl(\boldsymbol{n}).\end{aligned} \tag{C1}$$

Here, k_m and $|\boldsymbol{x}|$ are assumed to be very small numbers, which facilitates the following transformation:

$$(\Delta N_x)_{ijql} = \frac{1}{4w^3\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{iE_{qm}E_{jm}n_in_l}{\rho c_m^2}\int_0^{\infty}\frac{k_m^2\sin(k_m\tau)}{\tau}d\tau dl(\boldsymbol{n}). \tag{C2}$$

Because

$$\begin{aligned}\int_0^\infty \frac{\sin(k_m\tau)}{\tau}d\tau &= (\int_0^1+\int_1^\infty)\frac{\sin(k_m\tau)}{\tau}d\tau\\ &= k_m\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+1)(2i+1)!}-\int_1^\infty\frac{1}{k_m\tau}d\cos(k_m\tau)\\ &= k_m\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+1)(2i+1)!}+\frac{\cos k_m-1}{k_m}\\ &= k_m\Big[\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+1)(2i+1)!}-\frac{1}{2}\Big],\end{aligned}\tag{C3}$$

we finally arrive at the conclusion:

$$(\Delta N_x)_{ijql}=\frac{1}{4\pi^2}\sum_{m=1}^{3}\oint_{|\boldsymbol{n}|=1}\frac{E_{qm}E_{jm}n_in_l}{\rho c_m^5}\left[\sum_{i=0}^{\infty}(-1)^i\frac{1}{(2i+1)(2i+1)!}-\frac{1}{2}\right]dl(\boldsymbol{n}).\tag{C4}$$

The other equations are obtained in exactly the same way.